# The Cauchy integral formula and consequences.

Sunday, January 31, 2021 4:49 PM

**Theorem** (Cauchy integral formula).

**Let** $f \in A(B(z_0, r))$, $\gamma \subset B(z_0, r)$ - closed curve, $z \in B(z_0, r) \setminus \gamma$.

**Then**

$$\boxed{n(\gamma, z) \cdot f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{w - z}\, dw.}$$

**Proof.** Consider the function

$$F(w) := \frac{f(w) - f(z)}{w - z} \text{ in } B(z_0, r) \setminus \{z\}. \quad F(w) \in A(B(z_0, r) \setminus \{z\}),$$

$\lim_{w \to z} (w - z) F(w) = \lim_{w \to z} (f(w) - f(z)) = 0$. So, by Cauchy Theorem,

$$\int_\gamma F(w)\, dw = 0.$$

Or $$\int_\gamma \left( \frac{f(w)}{w - z} - \frac{f(z)}{w - z} \right) dw = 0$$

$$\boxed{\int_\gamma \frac{f(w)}{w - z}\, dw = f(z) \int_\gamma \frac{dw}{w - z} = f(z)\, n(\gamma, z)\, 2\pi i}$$

**Remark.** The same proof works if for some $z_1, \dots, z_n \neq z$, $f \in A(B(z_0, r) \setminus \{z_1, \dots, z_n\})$, $\lim_{z \to z_i} f(z)(z - z_i) = 0 \ \forall i$.

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**Important case:** $\gamma$ is a (piecewise smooth) Jordan curve oriented counterclockwise. $z$ — inside $\gamma$.

Then
$$\boxed{f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{w - z}\, dw.}$$

**Theorem.** Let $f \in A(D)$ ($D$ is a region).

Then $f$ is infinitely differentiable $\forall z \in D$.

Moreover, for any $z_0 \in D$, and $|z - z_0| < \operatorname{dist}(z_0, \partial D)$, we have

$$f(z) = \sum a_n (z - z_0)^n, \quad a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_{C_r} \frac{f(w)}{(w - z_0)^{n+1}}\, dz.$$

for any $r < \operatorname{dist}(z_0, \partial D)$, $C_r = \{z_0 + re^{it}\}$.

The series converges uniformly in $B(z_0, r)$ for any

$r < \text{dist}(z_0, \partial D)$ (converges locally uniformly).

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \ldots + \frac{f^{(n-1)}(z_0)}{(n-1)!}(z - z_0)^{n-1} + (z - z_0)^n f_n(z),$$

$$f_n(z) := \frac{1}{2\pi i}\oint_{C_r} \frac{f(z)\, dw}{(w-a)^n (w-z)}$$

(Taylor polynomial with Cauchy remainder).

where $|z - z_0| < r < \text{dist}(z_0, \partial D)$

Proof. Fix $z_0 \in D$. Consider $C_r = \{z_0 + re^{it}\}$.

$n(C_r, z_0) = 1$. So $\forall z \in B(z_0, r)$ we have

$$f(z) = \frac{1}{2\pi i}\oint_{C_r} \frac{f(w)}{w - z}\, dw$$ — Cauchy integral of $\frac{f(w)}{2\pi i}$ !

All we need to use is The Cauchy Integral Theorem.

This includes independence of all integrals on $r < \text{dist}(z_0, \partial D)$ - they represent the same quantity at $z_0$! ∎

Some consequences:

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1) Morera Theorem.

Let $f$ be continuous in a region $\Omega$.

Assume that for any $z_0 \in \Omega$ $\exists B(z_0, r) \subset \Omega$ with the